Spherical Waves

The utility of thinking of $\vec{k}$ as a ``ray'' becomes even more obvious when we get away from plane waves and start thinking of waves with *curved* wavefronts. The simplest such wave is the type that is emitted when a pebble is tossed into a still pool - an example of the ``point source'' that radiates waves isotropically in all directions. The wavefronts are then *circles* in two dimensions (the surface of the pool) or *spheres* in three dimensions (as for sound waves) separated by one wavelength λ and heading outward from the source at the propagation velocity c. In this case the ``rays'' $k$ point along the radius vector $\hat{r}$ from the source at any position and we can once again write down a rather simple formula for the ``wave function'' (displacement A as a function of position) that depends only on the time t and the *scalar* distance r from the source.

A plausible first guess would be just $A(x,t) \; = \; A_0 \; e^{i(kr-\omega t)}$, but this cannot be right! Why not? Because it violates energy conservation. The energy density stored in a wave is proportional to the square of its amplitude; in the trial solution above, the amplitude of the outgoing spherical wavefront is constant as a function or r, but the *area* of that wavefront increases as r^2. Thus the energy in the wavefront increases as r^2? I think not. We can get rid of this effect by just dividing the amplitude by r (which divides the energy density by r^2). Thus a trial solution is

$$\boxed{A(x,t) \; = \; A_0 \, \frac{e^{i(kr-\omega t)}}{r}.} \tag{14.38}$$

which is, as usual, correct.[14.11]The factor of $1/r$ accounts for the conservation of energy in the outgoing wave: since the spherical ``wave front'' distributes the wave's energy over a surface area $4\pi r^2$ and the flux of energy per unit area through a spherical surface of radius r is proportional to the *square* of the wave amplitude at that radius, the integral of $|f|^2$ over the entire sphere (*i.e.* the total outgoing *power*) is independent of r, as it must be.

We won't use this equation for anything right now, but it is interesting to know that it does accurately describe an outgoing[14.12]spherical wave.

The perceptive reader will have noticed by now that Eq. (38) is not a solution to the WAVE EQUATION as represented in one dimension by Eq. (10). That is hardly surprising, since the spherical wave solution is an intrinsically 3-dimensional beast; what happened to y and z? The correct *vector* form of the WAVE EQUATION is

$$\boxed{\frac{\partial^2 A}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0} \tag{14.39}$$

where the LAPLACIAN operator ∇^2 can be expressed in Cartesian[14.13]coordinates (x, y, z) as[14.14]

$$\nabla^2 \; = \; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{14.40}$$

With a little patient effort you can show that Eq. (38) does indeed satisfy Eq. (39), if you remember that $r = \sqrt{x^2 + y^2 + z^2}$. Or you can just take my word for it . . . .

Footnotes

. . . correct.[14.11]
: I should probably show you a few wrong guesses first, just to avoid giving the false impression that we always guess right the first time in Physics; but it would use up a lot of space for little purpose; and besides, ``knowing the answer'' is always the most powerful problem-solving technique!

. . . outgoing[14.12]
: One can also have ``incoming'' spherical waves, for which Eq. (38) becomes

$$A(x,t) \; = \; A_0 \, \frac{e^{i(kr+\omega t)}}{r}.$$

. . . Cartesian[14.13]
: The LAPLACIAN operator can also be represented in other coordinate systems such as spherical (r, θ, ϕ) or cylindrical (ρ, θ, z) coordinates, but I won't get carried away here.

. . . as[14.14]
: The LAPLACIAN operator can also be thought of as the inner (scalar or ``dot'') product of the GRADIENT operator $\vec{\nabla}$ with itself: $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$, where

$$\vec{\nabla} \; = \; \hat{\imath} \frac{\partial}{\partial x} \; + \; \hat{\jmath} \frac{\partial}{\partial y} \; + \; \hat{k} \frac{\partial}{\partial z}$$

in Cartesian coordinates. This VECTOR CALCULUS stuff is really elegant - you should check it out sometime - but it is usually regarded to be beyond the scope of an introductory presentation like this.

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Next: Electromagnetic Waves **Up:** WAVES **Previous:** Sound Waves

Jess H. Brewer 2002-03-26